Logarithmic version of the Milnor formula : research announcement (Algebraic Number Theory and Related Topics 2014)

AUTHOR(S):

Yang, Enlin

RIMS Kôkyûroku Bessatsu
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By

Enlin Yang*

Abstract

This is an announcement of the original paper [10], devoted to studying the Milnor formula with tamely ramified sheaves. In [10], we proposed a logarithmic version of the Milnor formula and proved this formula in the geometric case.

§ 1. Milnor formula

Let S be a regular scheme purely of dimension 1 and s a closed point of S with perfect residue field. Let (X, D) be a simple normal crossing pair, i.e., X is a regular scheme and D is a simple normal crossing divisor on X. Let $f\colon X \to S$ be a flat morphism of finite type. Let $x_0 \in D$ be a closed point of X such that $f(x_0) = s$. Assume that x_0 is a unique isolated log-singular point of f (with respect to the divisor D), i.e., $f|_{X-\{x_0\}}\colon X - \{x_0\} \to S$ is smooth and that $D - \{x_0\}$ is a divisor on $X - \{x_0\}$ with simple normal crossings relatively to S. Let ℓ be a prime number which is invertible in the residue field of S. Let $\mathcal{F}$ be a locally constant and constructible sheaf of $\mathbb{F}_\ell$-vector spaces on $U = X - D$. Assume that $\mathcal{F}$ is tamely ramified along D. Let $j\colon U \to X$ be the open immersion.

Since f has a unique isolated log-singularity at a closed point $x_0 \in D$, the support of the following coherent $\mathcal{O}_X$-module

$$\mathrm{T}^{\log}_{(\mathrm{X},\mathrm{D})/\mathrm{S}} = \mathcal{E}xt^1_{\mathcal{O}_X}\left(\Omega^1_{X/S}(\log D), \mathcal{O}_X\right) \tag{1.1}$$

is contained in $\{x_0\}$. Hence it is of finite length at x_0.

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*Department of Mathematical Sciences, Tsinghua University, Beijing, P. R. China 100084.
e-mail: `yangenlin0727@126.com,yangel10@mails.tsinghua.edu.cn`

Moreover, let $\overline{s} \to S$ be a geometric point of S with image s. Let $S_{(\overline{s})}$ be the spectrum of strict henselization of S at $\overline{s}$ and $X_{\overline{s}} = X \times_S S_{(\overline{s})}$. For the complex of vanishing cycles $\mathbf{R}\Phi(j_!\mathcal{F})$ on $X_{\overline{s}}$, a well known result ([1], Lemme 2.1.11 and Théorème 2.4.2) says that the cohomology groups $\mathbf{R}^i\Phi(j_!\mathcal{F})$ $(i \geq 0)$ of the complex of vanishing cycles $\mathbf{R}\Phi(j_!\mathcal{F})$ are sheaves of $\mathbb{F}_\ell$-vector spaces of finite dimension and are concentrated at $\overline{x}_0$. Notice that, since we assume that the residue field of S at s is perfect, the Swan conductor is defined (see [2, 8] for more details). Hence we can define the total dimension of $\mathbf{R}\Phi(j_!\mathcal{F})$ at x_0:

$$\text{dimtot } \mathbf{R}\Phi_{\overline{x}_0}(j_!\mathcal{F})\colon = \sum_{i\geq 0}(-1)^i(\dim + \text{Swan})\ \mathbf{R}^i\Phi_{\overline{x}_0}(j_!\mathcal{F}). \tag{1.2}$$

Then we conjecture that:

Conjecture 1.1 (Logarithmic Milnor Formula)**.** Under the above assumptions, we have

$$(-1)^n\text{dimtot } \mathbf{R}\Phi_{\overline{x}_0}(j_!\mathcal{F}) = \text{rank } \mathcal{F} \cdot \text{length}_{\mathcal{O}_{X,x_0}} \left\{\mathcal{E}xt^1_{\mathcal{O}_X}\left(\Omega^1_{X/S}(\log D), \mathcal{O}_X\right)\right\}_{x_0}, \tag{1.3}$$

where $n = \dim X - \dim S$ is the relative dimension of $f\colon X \to S$.

If D is empty and $\mathcal{F} = \mathbb{F}_\ell$, the above formula gives the classical Milnor formula (see [2]), which says that

$$(-1)^n\text{dimtot } \mathbf{R}\Phi_{\overline{x}_0}(\mathbb{F}_\ell) = \text{length}_{\mathcal{O}_{X,x_0}} \left\{\mathcal{E}xt^1_{\mathcal{O}_X}\left(\Omega^1_{X/S}, \mathcal{O}_X\right)\right\}_{x_0}. \tag{1.4}$$

The classical Milnor formula is true in the following cases (see [2, 5]):

- $n = 0, 1, 2$
- $X \to S$ has an ordinary quadratic singularity at x_0.
- S is of equal-characteristic.

In the geometric case where S is a scheme over an algebraically closed field of characteristic $p > 0$, the classical Milnor formula was proved by P. Deligne in [2]. In [5], F. Orgogozo showed that the conductor formula of Bloch implies the classical Milnor formula. In [4], K. Kato and T. Saito showed that the conductor formula is a consequence of an embedded resolution in a strong sense for the reduced closed fiber. Hence, the classical Milnor formula is true if we assume an embedded resolution. Consequently, the classical Milnor formula is true if the relative dimension is two. Recently, using Radon transform, T. Saito proved a Milnor formula with coefficient sheaf even for a normal surface in [6].

§ 2. Main theorem

In the paper [10], we proved that:

Theorem 2.1. *If S is of equal-characteristic, then the logarithmic Milnor formula is true.*

Here is a brief explanation of my idea how to prove this formula. We first prove a logarithmic refinement of Elkik's Lemma ([3], Lemme 2, p. 561):

Lemma 2.2. *Let (A, I) be a henselian pair, $B = A[X_1, \cdots, X_N]/J$ be an A-algebra of finite presentation such that $J = (f_1, \cdots, f_q)$ with $f_i \in A[X_1, \cdots, X_N]$. Let M be a non-negative integer such that $M \leq N$. Let Δ be the ideal of $A[X_1, \cdots, X_N]$ generated by order-q-minors of the logarithmic Jacobian matrix*

$$\mathrm{Jac}^{\log} = \begin{pmatrix} X_1 \frac{\partial f_1}{\partial X_1} & \cdots & X_M \frac{\partial f_1}{\partial X_M} & \frac{\partial f_1}{\partial X_{M+1}} & \cdots & \frac{\partial f_1}{\partial X_N} \\ X_1 \frac{\partial f_2}{\partial X_1} & \cdots & X_M \frac{\partial f_2}{\partial X_M} & \frac{\partial f_2}{\partial X_{M+1}} & \cdots & \frac{\partial f_2}{\partial X_N} \\ \vdots & & \vdots & \vdots & & \vdots \\ X_1 \frac{\partial f_q}{\partial X_1} & \cdots & X_M \frac{\partial f_q}{\partial X_M} & \frac{\partial f_q}{\partial X_{M+1}} & \cdots & \frac{\partial f_q}{\partial X_N} \end{pmatrix}.$$

For any $\mathbf{x} = (x_1, \cdots, x_N) \in A^N$, we define a homomorphism $\psi_{\mathbf{x}} \colon A[X_1, \cdots, X_N] \to A$ by mapping X_i to x_i. For any ideal $\mathcal{E}$ of $A[X_1, \cdots, X_N]$, the value $\mathcal{E}(\mathbf{x})$ of $\mathcal{E}$ at $\mathbf{x}$ is defined to be the image $\psi_{\mathbf{x}}(\mathcal{E})$. It is easy to see that $\mathcal{E}(\mathbf{x})$ is an ideal of A.

Let n and h be two integers such that $n > 2h$ and $\mathbf{a} = (a_1, \cdots, a_N) \in A^N$ such that

$$J(\mathbf{a}) \subset I^n \quad \textit{and} \quad I^h \subset \Delta(\mathbf{a}),$$

where the ideal $J(\mathbf{a}) = (f_1(\mathbf{a}), \cdots, f_q(\mathbf{a})) \subset I$ (resp. $\Delta(\mathbf{a})$) is the value of J (resp. Δ) at $\mathbf{a}$. Then there exists an element $\mathbf{b} = (b_1, \cdots, b_N) \in A^N$ such that the ideal $J(\mathbf{b})$ (value of J at $\mathbf{b}$) is zero and

$$b_r = a_r(1 + \epsilon_r), \qquad \epsilon_r \equiv 0 \mod I^{n-h}, \ \textit{for } r = 1, 2, \cdots, M \tag{2.1}$$

$$b_s \equiv a_s \mod I^{n-h}, \ \textit{for } s = M+1, M+2, \cdots, N. \tag{2.2}$$

Using this logarithmic refinement of Elkik's Lemma, we can deform a morphism to a curve. Then by a suggestion of Professor A. Abbes, we apply a result of I. Vidal ([9], Corollaire 3.4) by constructing a compactification. Then we can reduce the proof to the case where $\mathcal{F}$ is equal to $\mathbb{F}_\ell$. At last, the logarithmic Milnor formula in the geometric case is derived from P. Deligne's result [2].

§ 3. Characteristic cycle

Theorem 2.1 can also be interpreted in terms of characteristic cycle. From this point of view, Theorem 2.1 is a special case of a Milnor type formula which is conjectured by

P. Deligne (see [6]). P. Deligne's conjecture says that the total dimension of the space of vanishing cycles at an isolated characteristic point can be computed as an intersection number.

Let k be an algebraically closed field of characteristic $p > 0$. Let S be a smooth curve over k and (X, D) a simple normal crossing pair over k. Let $f\colon X \to S$ be a flat morphism of finite type such that a closed point $x_0 \in D$ is a unique isolated log-singular point of f. Let T^*S (resp. T^*X) be the cotangent bundle of S (resp. X). We have an induced morphism $T^*S \times_S X \to T^*X$ on vector bundles. We choose a local coordinate t of S on an open neighborhood S' of $s = f(x_0)$. When replacing S (resp. X) by S' (resp. $X' = X \times_S S'$), the values on both sides of (1.3) do not change. We may therefore assume that the local coordinate t is defined on S. Let $S \to T^*S$ be the section of $T^*S \to S$ defined by dt. By base change, we obtain a section $dt\colon X \to T^*S \times_S X$. Let $df\colon X \to T^*S \times_S X \to T^*X$ be the composition of the section $dt\colon X \to T^*S \times_S X$ with $T^*S \times_S X \to T^*X$.

For a regular immersion $g\colon X \to P$ of schemes, the conormal bundle $T^*_X P$ is the vector bundle over X defined by the symmetric algebra $S^\bullet(\mathcal{N}_{X/P})^\vee$ where $\mathcal{N}_{X/P} = \mathcal{I}_X/\mathcal{I}_X^2$ is the conormal sheaf and $\mathcal{I}_X = \ker(\mathcal{O}_P \to g_*\mathcal{O}_X)$ is the ideal sheaf of X in P. Let $D_1, \cdots, D_r$ be the irreducible components of D. For any subset $I \subset \{1, \cdots, r\}$, let $D_I = \cap_{i \in I} D_i$ with $D_\emptyset = X$. We define $T^*_{D_I}X \subset T^*X$ to be the conormal bundle associated to the regular immersion $D_I \hookrightarrow X$.

Let ℓ be a prime number distinct from p. Let $\mathcal{F}$ be a locally constant and constructible sheaf of $\mathbb{F}_\ell$-modules on $U = X - D$. Assume that $\mathcal{F}$ is tamely ramified along D. Denote by $j\colon U \to X$ the open immersion. The characteristic cycle $\mathrm{Char}(j_!\mathcal{F})$ is defined by (see [6])

$$\mathrm{Char}(j_!\mathcal{F}) = (-1)^m \cdot \sum_{I \subset \{1, \cdots, r\}} \mathrm{rank}\, \mathcal{F} \cdot [T^*_{D_I}X], \tag{3.1}$$

where $m = \dim X$. Using Serre's tor formula, we can prove that (see [10]) the logarithmic Milnor number $\mu^{\log} = \mathrm{length}_{\mathcal{O}_{X,x_0}} \left\{ \mathcal{E}xt^1_{\mathcal{O}_X}\left(\Omega^1_{X/S}(\log D), \mathcal{O}_X\right)\right\}_{x_0}$ is equal to the intersection number $\left(\sum_I [T^*_{D_I}X], [df(X)]\right)_{T^*X, x_0}$. Hence by Theorem 2.1, we have the following corollary.

Corollary 3.1. *Under the conditions above, we have*

$$-\mathrm{dimtot}\ \mathbf{R}\Phi_{\overline{x}_0}(j_!\mathcal{F}) = (\mathrm{Char}(j_!\mathcal{F}), [df(X)])_{T^*X, x_0}. \tag{3.2}$$

If $\mathcal{F}$ is not tame, the definition of characteristic cycle of $\mathcal{F}$ is very complicated. Recently, T. Saito proved a much more general result about Milnor type formula (for more details, see his manuscript [7]).

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